Lectures 13-14

Tuesday, October 29, 2019 12:13 PM

<u>Recall</u>. Assume $u, v: G \to \mathbb{R}$ have cont. partial derivatives (aka $\mathcal{C}^1$)
<u>then</u>: $f = u + iv : G \to \mathbb{C}$ is analytic $\iff$ $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ (Cauchy-Riemann or CR) in G

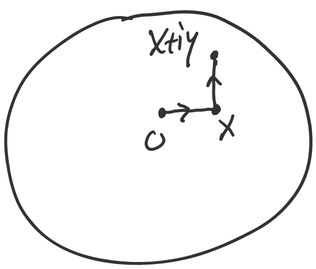
<u>Def</u>. A function u is <u>harmonic</u> in G if $u \in \mathcal{C}^2$ and
$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \left(= u_{xx} + u_{yy}\right) = 0 \text{ in } G.$$

<u>Rem.</u>
- If u, v are $\mathcal{C}^2$ and satisfy CR, then since mixed part. der. commute
$$u_{xx} = v_{yx} = v_{xy} = -u_{yy}\text{; i.e. } u_{xx} + u_{yy} = 0\text{, so } u \text{ is harmonic}$$
- A similar computation shows v is harmonic.

<u>Def</u>. If u, v are harmonic in G and $f = u + iv$ is analytic (i.e. satisfy CR), then v is <u>harmonic conjugate</u> of u.

<u>Prop 1</u>. Assume v_1, v_2 are both harmonic conjugates of u in G, and G is connected. Then $\exists$ constant $C \in \mathbb{R}$ s.t. $v_1 - v_2 = C$.

<u>Pf</u>. First, fix $B(a,r) \subseteq G$. For simplicity of notation and WLOG: $a = 0$.



For any harm. conj. v, $\begin{cases} v_x = -u_y \\ v_y = u_x \end{cases} \Rightarrow$

$$v(x,y) - v(0,0) = v(x,y) - v(x,0) + v(x,0) - v(0,0) = \{\text{Fund. Thm. of Calculus}\}$$
$$= \int_0^y v_y(x,t)\,dt + \int_0^x v_x(s,0)\,ds = \{CR\}$$
$$= \int_0^y u_x(x,t)\,dt - \int_0^x u_y(s,0)\,ds$$

Thus, $v(x,y) = \int_0^y u_x(x,t)\,dt - \int_0^x u_y(s,0)\,ds + v(0,0)$.

$\Rightarrow$ in $B($ [illegible] $) \subseteq G$ [illegible] $= C$

Thus, $v(x,y) = \int_0 u_x(x, \dots \int_0 v$

$\Rightarrow$ in $B(a,r) \subseteq G$, $v_1(x,y) - v_2(x,y) = v_1(a) - v_2(a) = C$

Let $A := \{z = x+iy \in G : v_1(x,y) - v_2(x,y) = C\}$. Then, $A \neq \emptyset$ and

Claim 1. A is open.

Let $a \in A$, and choose $B(a,r) \subseteq G$. By same arg. as above

$v_1(x,y) - v_2(x,y) = v_1(a) - v_2(a)$, $\forall x+iy \in B(a,r)$. But

$v_1(a) - v_2(a) = C$ by assumption $\Rightarrow$ $B(a,r) \subseteq A \Rightarrow A$ open.

Claim 2. A is closed.

Let b be limit point of A and $\{a_n\}$ seq. in A s.t. $a_n \to b$.

Then, $C = v_1(a_n) - v_2(a_n) \longrightarrow v_1(b) - v_2(b)$ by cont. of v_1, v_2.

But then $b \in A$. $\Rightarrow$ A closed since it contains all of its limit points.

Thm 1. Assume G is either $B(a,r)$ or $\mathbb{C}$. Then, every harmonic $u : G \to \mathbb{R}$ has a uniqe harmonic conjugate up to additive constant.

Pf. The part about uniqe up to additive constant is the content of Prop 1 (for any connected G).

The existence follows from the proof of Prop 1.

In $B(a,r)$ or $\mathbb{C}$, we can define v by (WLOG: $a=0$)

$$v(x,y) = \int_0^y u_x(x,t)\,dt - \int_0^x u_y(s,0)\,ds.$$

By FTC, $v_y(x,y) = u_x(x,y)$.

Also, $v_x(x,y) = \int_0^y u_{xx}(x,t)\,dt - u_y(x,0) = \{\Delta u = 0\}$

$\int^y \dots = \{FTC\}$

$$= -\int_0^y u_{yy}(x,t)\,dt - u_y(x,0) = \{FTC\}$$

$$= -\left(u_y(x,y) - u_y(x,0)\right) - u_y(x,0)$$

$$= -u_y(x,y).$$

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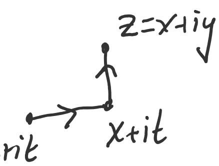
<u>Rem.</u> • Pf requires $\exists\, a \in G$ s.t. 

this figure is contained in G for all $z = x+iy \in G$.

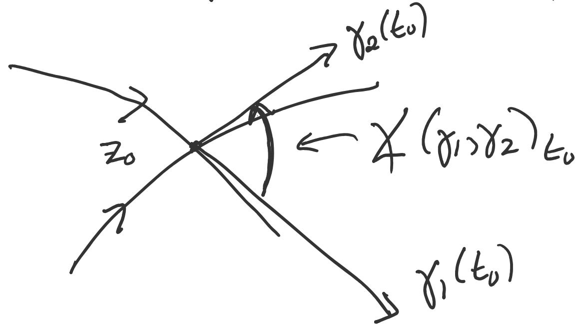
• In $G = \mathbb{C}\setminus\{0\}$, $u = \log|z|^2$ is harmonic (Why?) but if v were harm. conj., then $f = u+iv = \log z + C$, for some analytic branch of $\log z$ in G, but as we have seen, there is no analytic branch of $\log z$ in $\mathbb{C}\setminus\{0\}$.

<u>Analytic functions as mappings.</u>

<u>Def.</u> ① A <u>path</u> (or curve) in $G \subseteq \mathbb{C}$ (region) is a cont. map $\gamma : [a,b] \to G$. γ is $(\mathcal{C}^1$-)<u>smooth</u> if γ' exists and is cont. on $[a,b]$. γ is <u>piecewise smooth</u> if $[a,b] = \bigcup_{k=1}^{n} [a_{k-1}, a_k]$ w/ $a_0 = a$, $a_n = b$ and γ smooth on each $[a_{n-1}, a_n]$.

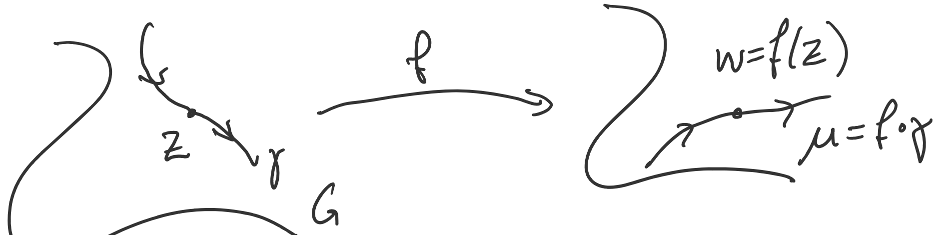
② If $\gamma_1, \gamma_2 : [a,b] \to G$ are smooth paths and for some $t_0 \in (a,b)$ $\gamma_1(t_0) = \gamma_2(t_0) = z_0$, and $\gamma_1'(t_0) \neq 0$; $\gamma_2'(t_0) \neq 0$, then the <u>angle between</u> γ_1 and γ_2 is $\measuredangle(\gamma_1,\gamma_2)_{t_0} \in [-\pi, \pi]$

the angle between γ_1 and γ_2 is $\measuredangle(\gamma_1,\gamma_2)_{t_0} \in [-\pi,\pi]$



*Note on Conway's def.

Note: • If $f: G \to \mathbb{C}$ is a complex function, we can view it as a map



• If $f = u+iv$ w/ $u,v \in \mathcal{C}'$, f is conformal (or angle preserving) at z_0 if $\forall$ smooth paths $\gamma_1,\gamma_2 : [a,b] \to G$ as in ② above, the angle between γ_1 and γ_2 at t_0 equals the angle between $\mu_1 := f\circ\gamma_1$ and $\mu_2 = f\circ\gamma_2$ at t_0 (meaning also $\mu_1'(t_0) \neq 0$, $\mu_2'(t_0) \neq 0$).

Thm 2. If f is analytic in G and $f' \neq 0$ in $G' \subseteq G$, then f is conformal in G'.

Pf. Let γ_1,γ_2 be as above w/ $z_0 = \gamma_1(t_0) = \gamma_2(t_0) \in G'$.
Since f is analytic, chain rule $\Rightarrow$

$\mu_1'(t_0) = f'(z_0)\gamma_1'(t_0)$; $\mu_2'(t_0) = f'(z_0)\gamma_2'(t)$

and $f'(z_0) \neq 0$. Thus, $\mu_1'(t_0)$, $\mu_2'(t_0)$ are obtained from $\gamma_1'(t_0)$, $\gamma_2'(t_0)$ by multiplication by same

and $f'(z_0)\,\ldots$
from $\gamma_1'(t_0), \gamma_2'(t_0)$ by multiplication by same nonzero number $f'(z_0)$ we immediately see $\measuredangle(\gamma_1, \gamma_2) = \measuredangle(\mu_1, \mu_2) \Rightarrow f$ is conformal at z_0.

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Möbius transformations.

Def. • A <u>linear fractional transformations</u> is a map $S: \mathbb{C}_\infty \to \mathbb{C}_\infty$ given by $S(z) = \dfrac{az+b}{cz+d}$, $S(-d/c) = \infty$, $S(\infty) = a/c$.

• $S(z)$ is a <u>Möbius transformation</u> if $ad-bc \neq 0$.

<u>Basic Props of Möbius trans.</u>

1. $S: \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a homeomorphism (conf. + bijective), and an analytic function on $\mathbb{C} \setminus \{-d/c\}$.
2. The inverse is $S^{-1}(z) = \dfrac{dz-b}{-cz+a}$.
3. If $\underbrace{S(z)=z}_{\text{equation}}$ has 3 solutions $z_1, z_2, z_3 \in \mathbb{C}_\infty$ (fixed pts), then in fact $S(z)$ is identity map ($S(z)=z$).

<u>Sketch of pf.</u> (1) Analyticity is clear. Cont. on $\mathbb{C}_\infty$ is <u>Ex.</u> Bijectivity follows from (2).

(2) Solve equation $w = \dfrac{az+b}{cz+d}$ for z.

(3) Consider the equation

$$z = \frac{az+b}{cz+d} \Rightarrow cz^2 + (d-a)z - b = 0. \quad (*)$$

Either $c=0$, $d=a$, $b=0 \Rightarrow S(z)=z$ or
$(*)$ is non-trivial quadratic eq. $\Rightarrow$ can at most

Either $c=0$, $a=d$, $b=0 \Rightarrow S(z)$

$(\ast)$ is non-trivial quadratic eq. $\Rightarrow$ can at most have 2 distinct solutions.

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Prop 2. Given $z_1, z_2, z_3 \in \mathbb{C}_\infty$ $\exists$ unique Möbius S w/ $S(z_1)=1$, $S(z_2)=0$, $S(z_3)=\infty$.

Pf. Suppose $z_1, z_2, z_3 \in \mathbb{C}$ then

$$S(z) = \frac{z-z_2}{z-z_3} \cdot \frac{z_1-z_3}{z_1-z_2}.$$

If say $z_2=\infty$, then $S(z) = \dfrac{z_1-z_3}{z-z_3}$, etc.

$\boxed{/\!/}$

Thm 3. Given $z_1, z_2, z_3, w_1, w_2, w_3 \in \mathbb{C}_\infty$ $\exists$ unique S w/ $S(z_j)=w_j$ for $j=1,2,3$.

Pf. Let T_1, T_2 be maps given by Prop 2 for (z_1, z_2, z_3) and (w_1, w_2, w_3) respectively. Then $S = T_2^{-1} \circ T_1$ does the trick. To see this S is unique, we note that if there are S_1, S_2 that sends (z_1, z_2, z_3) to (w_1, w_2, w_3), then $S_2^{-1} \circ S_1$ has 3 fixed points (namely z_1, z_2, z_3) and, hence, $S_2^{-1} \circ S_1 = I$ or

$S_1 = S_2$.

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